

# MODULE 1: INTRODUCTION TO GRAPH THEORY

## Graph

A graph  $G=(V,E)$   $G = (V, E)$   $G=(V,E)$  consists of:

- $V \rightarrow$  set of **vertices (nodes)**
- $E \rightarrow$  set of **edges (links)** connecting vertices

Types:

- **Simple graph** – no loops, no multiple edges
  - **Multigraph** – multiple edges allowed
  - **Pseudograph** – loops allowed
- 

## Basic Terminology

- **Degree of a vertex:** Number of edges incident to it
  - **Pendant vertex:** Vertex of degree 1
  - **Isolated vertex:** Vertex of degree 0
  - **Loop:** Edge connecting a vertex to itself
  - **Adjacent vertices:** Vertices connected by an edge
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## Walk, Path, Circuit

- **Walk:** Sequence of vertices where edges may repeat
- **Trail:** Walk with no repeated edges
- **Path:** Walk with no repeated vertices
- **Circuit (Cycle):** Closed path (start = end)

★ *Exam tip:* Path  $\rightarrow$  no repetition, Walk  $\rightarrow$  repetition allowed

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## Connectedness

- A graph is **connected** if there is a path between **every pair of vertices**
  - If not connected  $\rightarrow$  **disconnected graph**
  - Each connected part is called a **component**
-

# Handshaking Lemma

**Statement:**

Sum of degrees of all vertices =  $2 \times$  number of edges

$$\sum \deg(v) = 2|E|$$

**Important result:**

- Number of vertices with **odd degree is always even**

✦ Very important for short questions

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# Isomorphism

Two graphs are **isomorphic** if:

- They have the same structure
- Only vertex labels differ

To check:

- Same number of vertices
  - Same number of edges
  - Same degree sequence
- 

# Subgraph

A graph formed from a **subset of vertices and edges** of a graph

Types:

- **Proper subgraph**
  - **Induced subgraph**
- 

# Reachability

- Vertex **v** is reachable from **u** if there exists a path from **u to v**
-

## Union and Intersection of Graphs

- **Union:** Combines vertices and edges of both graphs
  - **Intersection:** Common vertices and edges only
- 

# MODULE 2: GRAPH TRANSVERSAL

## Euler Graph

A graph is **Eulerian** if:

- It is connected
- **All vertices have even degree**

### Euler Path

- Exactly **two vertices** have odd degree

✦ Euler graph → covers every edge **exactly once**

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## Shortest Path Problem

Finding the **minimum distance path** between two vertices

Common algorithm:

- **Dijkstra's Algorithm**

Applications:

- GPS navigation
  - Network routing
- 

## Hamiltonian Graph

- Contains a **Hamiltonian cycle**
- Visits **every vertex exactly once**

✦ No simple test like Euler graphs

✦ Often solved by logic or conditions

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## Traveling Salesman Problem (TSP)

- Find the **shortest possible Hamiltonian cycle**
- NP-hard problem

Applications:

- Delivery routing
  - Scheduling
- 

## Bipartite Graph

Vertices divided into **two disjoint sets**

- Edges connect only between sets, not within

Key property:

- **No odd-length cycles**

Examples:

- Matching problems
  - Job assignment
- 

# MODULE 3: TREES

## Tree

A **tree** is a graph that is:

- Connected
- Acyclic (no cycles)

Properties:

- For **n vertices**, edges =  **$n - 1$**
  - Exactly **one path** between any two vertices
-

## Rooted Tree

- One vertex chosen as **root**
- Levels defined from root

Terms:

- Parent
  - Child
  - Leaf (degree 1)
- 

## Path Length in Rooted Trees

- Number of edges between root and a vertex
  - Used to measure **depth**
- 

## Spanning Tree

- A subgraph that:
  - Includes all vertices
  - Is a tree

✦ Graph can have **many spanning trees**

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## Fundamental Circuit

- Formed by adding one edge to a spanning tree
  - Exactly **one cycle** is created
- 

## Weighted Graph

Edges have weights (cost, distance, time)

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## Cut Set and Cut Vertex

- **Cut vertex:** Removing it disconnects the graph
- **Cut set:** Set of edges whose removal disconnects the graph

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## Fundamental Cut Set

- Cut set associated with a particular spanning tree edge
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## Minimum Spanning Tree (MST)

Spanning tree with **minimum total weight**

Algorithms:

- **Kruskal's Algorithm**
- **Prim's Algorithm**

Applications:

- Network design
  - Wiring systems
- 

# MODULE 4: DIRECTED & PLANAR GRAPHS

## Directed Graph (Digraph)

Edges have **direction**

Terms:

- **In-degree**
  - **Out-degree**
- 

## Directed Connectedness

- **Strongly connected:** Path exists in both directions
  - **Weakly connected:** Connected ignoring direction
-

## Directed Trees

- Exactly one root
  - All edges directed away from or toward root
- 

## Network Flow

Graph with:

- Source
  - Sink
  - Capacity on edges
- 

## Max Flow – Min Cut Theorem

Maximum flow from source to sink = Minimum cut capacity

✦ Very important theorem

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## Matrix Representation of Graph

1. **Adjacency Matrix**
2. **Incidence Matrix**

Used for:

- Computer representation
  - Algorithm implementation
- 

## Planar Graph

Can be drawn in a plane **without edge crossings**

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## Dual Graph

- **Geometric dual:** Formed using faces
- **Combinational dual:** Based on structure

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## Kuratowski's Theorem

A graph is **non-planar** if it contains:

- $K_5$  or  $K_{3,3}$  or
- $K_5$  or  $K_{3,3}$  as a subdivision

✦ High exam weight

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## Thickness and Crossing Number

- **Thickness:** Minimum number of planar graphs needed
  - **Crossing number:** Minimum edge crossings
- 

# MODULE 5: COMBINATORICS

## Partition

Ways of writing a number as a sum of positive integers

Example:

- $5 = 4+1 = 3+2 = 2+2+1$
- 

## Counting Functions

Number of functions between two sets

Types:

- One-to-one (injective)
  - Onto (surjective)
  - Bijective
- 

## Partitions into Odd or Unequal Parts

Important result:

Number of partitions into odd parts = number of partitions into distinct parts

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## Necklaces

Arrangement of beads considering **rotations**

Used with:

- **Burnside's Lemma**
  - **Polya's Theorem**
- 

## Euler's Totient Function $\phi(n)$

Number of integers  $\leq n$  that are **coprime** to  $n$

Example:

- $\phi(8) = 4$
- 

## Set of Symmetries

Includes:

- Rotations
- Reflections

Used for:

- Enumeration problems
- 

## Enumeration (Odd & Even Cases)

Counting distinct arrangements considering symmetry

# MODULE 1: INTRODUCTION TO GRAPH THEORY

## Q1. Define a graph and explain its types

**Answer:**

A **graph**  $G=(V,E)$  consists of a non-empty set of vertices  $V$  and a set of edges  $E$  connecting pairs of vertices.

**Types of graphs:**

1. **Simple graph** – no loops or parallel edges
  2. **Multigraph** – parallel edges allowed
  3. **Pseudograph** – loops allowed
  4. **Directed graph** – edges have direction
  5. **Weighted graph** – edges have weights
- 

## Q2. Define walk, path, trail, and circuit

**Answer:**

- **Walk:** A sequence of vertices where edges may repeat
  - **Trail:** Walk with no repeated edges
  - **Path:** Walk with no repeated vertices
  - **Circuit:** Closed path (start = end)
- 

## Q3. What is a connected graph?

**Answer:**

A graph is **connected** if there exists a path between **every pair of vertices**. If not, it is called **disconnected**, and each connected part is called a **component**.

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## Q4. State and prove Handshaking Lemma

**Answer:**

**Statement:**

In any undirected graph, the sum of degrees of all vertices equals twice the number of edges.

$$\sum \deg(v) = 2|E|$$

**Proof:**

Each edge contributes **2 degrees** (one at each end).  
Hence total degree count =  $2 \times 2 \times$  number of edges.

**Result:**

Number of vertices with odd degree is always even.

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## Q5. What is graph isomorphism?

**Answer:**

Two graphs are **isomorphic** if they have the same structure but different labels.

**Conditions:**

- Same number of vertices
  - Same number of edges
  - Same degree sequence
- 

## MODULE 2: GRAPH TRANSVERSAL

### Q6. Define Euler graph and Euler path

**Answer:**

- **Euler graph:** A connected graph in which **all vertices have even degree**.
  - **Euler path:** A path that uses **every edge exactly once**.
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### Q7. State Euler's theorem

**Answer:**

A connected graph is Eulerian **if and only if** every vertex has even degree.

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### Q8. Define Hamiltonian graph

**Answer:**

A graph is **Hamiltonian** if it contains a **Hamiltonian cycle**, i.e., a cycle that visits **every vertex exactly once**.

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## Q9. Explain Traveling Salesman Problem (TSP)

**Answer:**

TSP finds the **shortest Hamiltonian cycle** visiting every vertex exactly once and returning to the starting point.

It is an **NP-hard problem**.

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## Q10. Define bipartite graph

**Answer:**

A graph is **bipartite** if its vertices can be divided into two disjoint sets such that edges connect only vertices from different sets.

**Property:**

Bipartite graphs contain **no odd cycles**.

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## MODULE 3: TREES

### Q11. Define a tree and state its properties

**Answer:**

A **tree** is a connected graph with no cycles.

**Properties:**

- If a tree has  $n$  vertices, it has  $n-1$  edges
  - Exactly one path exists between any two vertices
  - Removing any edge disconnects the tree
- 

### Q12. Prove that a tree with $n$ vertices has $n-1$ edges

**Answer:**

A tree is minimally connected.

If it had fewer than  $n-1$  edges  $\rightarrow$  disconnected

If more than  $n-1$  edges  $\rightarrow$  cycle formed

Therefore, a tree has exactly  $n-1$  edges.

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### Q13. What is a rooted tree?

**Answer:**

A **rooted tree** is a tree in which one vertex is designated as the **root**. Edges are oriented away from or toward the root.

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### Q14. Define spanning tree

**Answer:**

A **spanning tree** is a subgraph that:

- Includes all vertices
  - Is connected
  - Contains no cycles
- 

### Q15. What is a minimum spanning tree?

**Answer:**

A **minimum spanning tree (MST)** is a spanning tree of a weighted graph with **minimum total edge weight**.

**Algorithms:**

- Kruskal's Algorithm
  - Prim's Algorithm
- 

## MODULE 4: DIRECTED & PLANAR GRAPHS

### Q16. Define directed graph

**Answer:**

A **directed graph (digraph)** has edges with direction.

- **In-degree:** Number of incoming edges
  - **Out-degree:** Number of outgoing edges
-

## Q17. Explain strong and weak connectedness

Answer:

- **Strongly connected:** Path exists in both directions between every pair
  - **Weakly connected:** Connected if direction is ignored
- 

## Q18. State Max-Flow Min-Cut theorem

Answer:

The maximum flow from source to sink equals the minimum cut capacity of the network.

---

## Q19. What is a planar graph?

Answer:

A graph is **planar** if it can be drawn in a plane **without edge crossings**.

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## Q20. State Kuratowski's theorem

Answer:

A graph is planar if and only if it does not contain a subdivision of  $K_5$  or  $K_{3,3}$ .

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## Q21. State Euler's formula for planar graphs

Answer:

For a connected planar graph:

$$V - E + F = 2$$

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# MODULE 5: COMBINATORICS

## Q22. Define partition of a number

**Answer:**

A **partition** of a number is a way of writing it as a sum of positive integers, order ignored.

Example:

$$5 = 3 + 2 = 2 + 2 + 1$$

---

## Q23. Define Euler's Totient function

**Answer:**

$\phi(n)$  is the number of integers  $\leq n$  that are **coprime to n**.

Example:

$$\phi(10) = 4 \rightarrow \{1, 3, 7, 9\}$$

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## Q24. State partition theorem

**Answer:**

The number of partitions of a number into odd parts equals the number of partitions into distinct parts.

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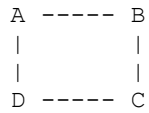
## Q25. State Burnside's Lemma

**Answer:**

The number of distinct arrangements equals the average number of arrangements fixed by group symmetries.

# MODULE 1: INTRODUCTION TO GRAPH THEORY

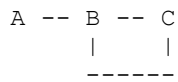
## 1. Simple Graph



*Caption:* A simple graph has no loops and no multiple edges.

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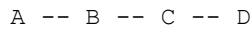
## 2. Walk



*Caption:* Walk allows repetition of vertices and edges.

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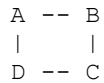
## 3. Path



*Caption:* Path has no repeated vertices.

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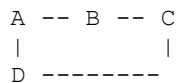
## 4. Circuit (Cycle)



*Caption:* A circuit is a closed path.

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## 5. Connected Graph



*Caption:* A graph is connected if every pair of vertices has a path.

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## 6. Disconnected Graph

A -- B      C -- D

*Caption:* A disconnected graph has more than one component.

---

## 7. Degree of a Vertex

```
      B
      |
A --- C --- D
      |
      E
```

*Caption:* Degree of vertex C is 4.

---

## 8. Subgraph

```
Original:      Subgraph:
A -- B -- C    A -- B
|              |
D -----
```

*Caption:* A subgraph is formed from a subset of vertices and edges.

---

## 9. Isomorphic Graphs

```
A -- B          1 -- 2
|              |
D -- C          4 -- 3
    ℜ
```

*Caption:* Isomorphic graphs have the same structure but different labels.

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# MODULE 2: GRAPH TRANSVERSAL

## 10. Euler Graph

```
A -- B
|     |
D -- C
```

*Caption:* Euler graph has all vertices of even degree.

---

## 11. Euler Path

A -- B -- C -- D

*Caption:* Euler path exists when exactly two vertices have odd degree.

---

## 12. Hamiltonian Cycle

A -- B  
|     |  
D -- C

*Caption:* Hamiltonian cycle visits every vertex exactly once.

---

## 13. Bipartite Graph

U-set            V-set  
A ----- 1  
B ----- 2  
C ----- 3

*Caption:* Vertices can be divided into two sets with no edges within a set.

---

## 14. Odd Cycle (Not Bipartite)

A -- B  
  \<   /  
   C

*Caption:* Presence of odd cycle means graph is not bipartite.

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# MODULE 3: TREES

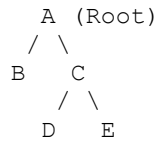
## 15. Tree

   A  
  /<  \  
B      C  
  /<  \  
  D   E

*Caption:* A tree is connected and has no cycles.

---

## 16. Rooted Tree



*Caption:* Rooted tree has a designated root vertex.

---

## 17. Unique Path in Tree

A -- B -- C -- D

*Caption:* Exactly one path exists between any two vertices in a tree.

---

## 18. Spanning Tree

Graph:	Spanning Tree:
A -- B	A -- B
\ /	
/ \	D -- C
D -- C	

*Caption:* A spanning tree includes all vertices with no cycles.

---

## 19. Cut Vertex

```
A -- B -- C
      |
      D
```

*Caption:* Removing B disconnects the graph.

---

## 20. Minimum Spanning Tree

```
A --2-- B
|         |
3         1
|         |
```

D --4-- C

*Caption:* MST has minimum total edge weight.

---

## MODULE 4: DIRECTED & PLANAR GRAPHS

### 21. Directed Graph

```
A ---> B ---> C
^           |
|-----|
```

*Caption:* Edges have direction in a directed graph.

---

### 22. Strongly Connected Digraph

```
A ---> B
^         |
|         v
D <--- C
```

*Caption:* Path exists in both directions between every pair.

---

### 23. Network Flow

```
(S) ---> A ---> (T)
 \       |
  ---> B ---
```

*Caption:* Flow moves from source to sink with capacities.

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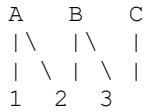
### 24. Planar Graph

```
A -- B
|    |
D -- C
```

*Caption:* Can be drawn without edge crossings.

---

### 25. Non-Planar Graph ( $K_{3,3}$ )



*Caption:*  $K_{3,3}$  is non-planar.

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## MODULE 5: COMBINATORICS

### 26. Partition of a Number

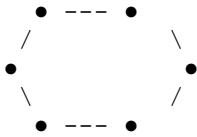
$$5 = 3 + 2$$

$$5 = 2 + 2 + 1$$

*Caption:* Partition writes a number as sum of positive integers.

---

### 27. Necklace (Rotation Symmetry)



*Caption:* Rotations give identical arrangements.

---

### 28. Symmetry (Reflection + Rotation)



*Caption:* Symmetry operations reduce distinct arrangements.

---

### 29. Euler's Totient Example

$$n = 10$$

$$\text{Coprime: } 1, 3, 7, 9$$

*Caption:*  $\phi(n)$  counts integers coprime to  $n$ .

# GRAPH THEORY & COMBINATORICS

## 50 IMPORTANT MCQs

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### Module 1: Basics of Graph Theory

**1. A graph with loops allowed but no parallel edges is called**

- A. Simple graph
- B. Multigraph
- C. Pseudograph
- D. Directed graph

**2. The degree of an isolated vertex is**

- A. 1
- B. 0
- C. -1
- D. 2

**3. In an undirected graph, each edge contributes how many degrees?**

- A. 1
- B. 2
- C. 0
- D. Depends on graph

**4. A walk in which edges are not repeated is called**

- A. Walk
- B. Path
- C. Trail
- D. Circuit

**5. A closed walk with no repeated edges is called**

- A. Path
- B. Circuit
- C. Trail
- D. Euler graph

**6. A graph is connected if**

- A. It has no isolated vertices
- B. There exists a path between every pair of vertices
- C. All vertices have equal degree
- D. It has cycles

**7. The Handshaking Lemma applies to**

- A. Directed graphs

- B. Undirected graphs
- C. Trees only
- D. Weighted graphs only

**8. Number of odd degree vertices in a graph is always**

- A. Odd
- B. Even
- C. Prime
- D. Zero

**9. Two graphs are isomorphic if they have**

- A. Same labels
- B. Same drawing
- C. Same structure
- D. Same number of faces

**10. A subgraph contains**

- A. All vertices and edges
- B. Only edges
- C. Subset of vertices and edges
- D. Only vertices

---

## **Module 2: Euler, Hamiltonian & Bipartite Graphs**

**11. A connected graph is Eulerian if**

- A. Exactly two vertices have odd degree
- B. All vertices have even degree
- C. All vertices have odd degree
- D. It has no cycles

**12. An Euler path exists if**

- A. All vertices are even degree
- B. Exactly two vertices are odd degree
- C. More than two vertices are odd degree
- D. Graph is complete

**13. Which graph problem visits each edge exactly once?**

- A. Hamiltonian
- B. Shortest path
- C. Euler
- D. Spanning tree

**14. A Hamiltonian cycle visits**

- A. Every edge once
- B. Every vertex once
- C. Some vertices
- D. Only even degree vertices

**15. Traveling Salesman Problem is based on**

- A. Euler path
- B. Bipartite matching
- C. Hamiltonian cycle
- D. Minimum cut

**16. A bipartite graph has vertex set divided into**

- A. Three sets
- B. Two disjoint sets
- C. Even and odd sets
- D. Complete sets

**17. A graph is bipartite if it contains**

- A. Even cycles only
- B. Odd cycles only
- C. No cycles
- D. Complete graph

**18. Which of the following graphs is always bipartite?**

- A. Cycle of length 3
- B. Cycle of length 5
- C. Cycle of length 4
- D. Complete graph  $K_5$

**19. Dijkstra's algorithm is used to find**

- A. Euler path
- B. Minimum spanning tree
- C. Shortest path
- D. Hamiltonian cycle

**20. Which problem is NP-hard?**

- A. Euler path
- B. TSP
- C. BFS
- D. Spanning tree

---

### **Module 3: Trees & Spanning Trees**

**21. A tree is a graph which is**

- A. Cyclic and connected
- B. Acyclic and connected
- C. Disconnected
- D. Complete

**22. A tree with  $n$  vertices has how many edges?**

- A.  $n$
- B.  $n + 1$

- C.  $n - 1$
- D.  $2n$

**23. In a tree, removing any edge will make the graph**

- A. Cyclic
- B. Complete
- C. Disconnected
- D. Directed

**24. A vertex of degree 1 in a tree is called**

- A. Root
- B. Leaf
- C. Cut vertex
- D. Parent

**25. A spanning tree of a graph**

- A. Has cycles
- B. Includes all vertices
- C. Has maximum edges
- D. Is always unique

**26. A graph can have how many spanning trees?**

- A. Zero
- B. One only
- C. Exactly two
- D. More than one

**27. Fundamental circuit is formed by**

- A. Removing an edge
- B. Adding one edge to spanning tree
- C. Removing a vertex
- D. Adding a vertex

**28. A cut vertex is a vertex whose removal**

- A. Reduces degree
- B. Adds cycles
- C. Disconnects the graph
- D. Makes graph complete

**29. Minimum spanning tree minimizes**

- A. Number of vertices
- B. Number of edges
- C. Total edge weight
- D. Degree of vertices

**30. Prim's algorithm is used for**

- A. Shortest path
- B. MST
- C. Euler circuit
- D. Hamiltonian cycle

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## Module 4: Directed & Planar Graphs

**31. In-degree of a vertex is**

- A. Total edges
- B. Outgoing edges
- C. Incoming edges
- D. Undirected edges

**32. A digraph is strongly connected if**

- A. Direction is ignored
- B. Path exists in one direction
- C. Path exists in both directions
- D. All degrees are equal

**33. Network flow graph must have**

- A. Only source
- B. Only sink
- C. Source and sink
- D. No vertices

**34. Max-Flow Min-Cut theorem states that**

- A. Max flow < Min cut
- B. Max flow > Min cut
- C. Max flow = Min cut
- D. No relation

**35. Adjacency matrix of a graph with n vertices is of order**

- A.  $n \times n$
- B.  $n \times e$
- C.  $e \times e$
- D.  $v \times e$

**36. A planar graph can be drawn**

- A. With minimum edges
- B. With crossings
- C. Without edge crossings
- D. Only using straight lines

**37. Euler's formula for planar graph is**

- A.  $V + E + F = 2$
- B.  $V - E + F = 2$
- C.  $V \times E = F$
- D.  $E - V = 2$

**38. Which graph is non-planar?**

- A.  $K_4$
- B.  $K_3$

- C.  $K_5$
- D. Tree

**39. Kuratowski's theorem involves**

- A. Trees
- B. Euler graphs
- C.  $K_5$  and  $K_{3,3}$
- D. Bipartite graphs

**40. Crossing number of a planar graph is**

- A. 1
  - B. 2
  - C. 0
  - D. Infinite
- 

## **Module 5: Combinatorics**

**41. A partition of a number ignores**

- A. Value
- B. Order
- C. Size
- D. Frequency

**42. Number of partitions into odd parts equals partitions into**

- A. Prime parts
- B. Even parts
- C. Distinct parts
- D. Equal parts

**43. A one-to-one function is also called**

- A. Surjective
- B. Injective
- C. Bijective
- D. Constant

**44. A bijective function is**

- A. One-to-one only
- B. Onto only
- C. Both one-to-one and onto
- D. Neither

**45. Euler's Totient function  $\phi(n)$  counts**

- A. Divisors
- B. Prime numbers
- C. Coprime numbers
- D. Factors

**46.  $\phi(7)$  is**

- A. 7
- B. 6
- C. 5
- D. 4

**47. Necklace problems use**

- A. Dijkstra's algorithm
- B. Polya's theorem
- C. Kruskal's algorithm
- D. BFS

**48. Burnside's lemma is used for**

- A. Sorting
- B. Searching
- C. Enumeration
- D. Optimization

**49. Symmetry group of a square has how many elements?**

- A. 2
- B. 4
- C. 6
- D. 8

**50. Enumeration problems consider**

- A. Order only
- B. Size only
- C. Symmetry
- D. Graph weight

---

## ANSWER KEY

- 1. C
- 2. B
- 3. B
- 4. C
- 5. B
- 6. B
- 7. B
- 8. B
- 9. C
- 10. C
- 11. B
- 12. B
- 13. C
- 14. B
- 15. C

16. B
17. A
18. C
19. C
20. B
21. B
22. C
23. C
24. B
25. B
26. D
27. B
28. C
29. C
30. B
31. C
32. C
33. C
34. C
35. A
36. C
37. B
38. C
39. C
40. C
41. B
42. C
43. B
44. C
45. C
46. B
47. B
48. C
49. D
50. C